Analysis of Irregular Shapes

Presented by

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Acknowledgement

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Q. Guo, J. Shao, F. Guo, and V.F. Ruiz, "Irregular Shape Symmetry Analysis—Part I:Theory," *IEEE Trans. Pattern Anal. Machine Intell.*, submitted for publication.

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Introduction

- Symmetry is one of the basic features of shapes and objects
  - studied in both computer vision and image processing fields

- Symmetrical descriptions of shapes and detection of symmetrical features of objects are useful in guiding:
  - shape matching, model-based object matching and object recognition

- However, the exact mathematical definition of symmetry is inadequate to describe and quantify the symmetries found in real objects and images.
  - often a binary concept: either an object is symmetric or it is not at all

Introduction

- Perfect or crisp symmetry is rare in nature
  - Most objects have only approximate or imperfect symmetries
  - To describe the imperfect symmetry quantitatively, one has to deal with approximate symmetries

- We propose a set of geometric symmetry measures:
  - used to quantitatively measure the “amount” of symmetry of graphs with arbitrary shapes
  - A mathematical framework based on group theory is developed for quantifying the degree of symmetry of graphs.
  - This framework provides a new way to analyse and characterise the irregularity of arbitrary shapes

Relationship between Regularity and Symmetry

- Regularity means binding, order and harmony
- Symmetry is one of the basic properties of nature
  - Mathematician Hermann Weyl, 1952
    - "symmetry = harmony of proportions."
    - "… symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole."

Relationship between Regularity and Symmetry

- We argue that one of the basic regularity features of shapes and graphs is symmetry:
  - No matter how complicated a geometric graph is, the basic regularity property of the graph in the transition from irregular shape to regular shape is its increasing “amount” of symmetry
  - Basic symmetries:
    - bilateral symmetry (reflection)
    - translational symmetry
    - rotational symmetry

Relationship between Regularity and Symmetry

- A number of researchers have tried to detect symmetry and quantify the degree of symmetry by using a symmetry measure

- Most of these symmetry measure and detection methods are concerned with the symmetrical relationships between two objects or the symmetry property of the various transforms of a given object

- The choice of symmetry measure, which is used to quantify the degree of symmetry, is often based on a distance function (or metric)

Relationship between Regularity and Symmetry

- Rarely, symmetry has been studied as a regularity attribute of the objects and shapes themselves quantitatively.
- We develop **fuzzy symmetry measures** based solely on geometrical operations to characterise the irregularity of graph with arbitrary shape.

Why Group Theory

1. **Miller (1972) showed that the symmetry properties of a given system may be summarised by specifying its symmetry group**

2. **Group theory, is a powerful tool in the investigation of geometric transformation (Miller, 1972)**

3. **It is easier to specify individual operation or location of the operation during the transformation process using group**

Why Group Theory

In other word

- Group theory provides the adequate mathematical language to define and introduce this new concept of symmetry analysis and continuous measure.

Shape Symmetry Analysis: Some Basics

- Bilateral symmetry and rotational symmetry are two common types of symmetries
  - A shape is said to be **bilateral symmetric** if it is invariant to reflection about a line (symmetric axis, passing through the centroid of the shape)
  - A shape is said to be **order *n* rotationally symmetric** if it is invariant under rotations of $2\pi/n$ radians about its centre of mass

Shape Symmetry Analysis: Some Basics

- **The difference between regular shape and irregular shape lies in the degree of symmetry possessed by the shape**

- **For regular shapes**:
  - There are two types of rotation-based symmetries, which correspond to finite groups of rotations about a centre O in plane geometry, given by Leonardo's table (in Weyl, 1952) as follows:

$$C_1,\ C_2,\ C_3,\ \cdots\ ;$$

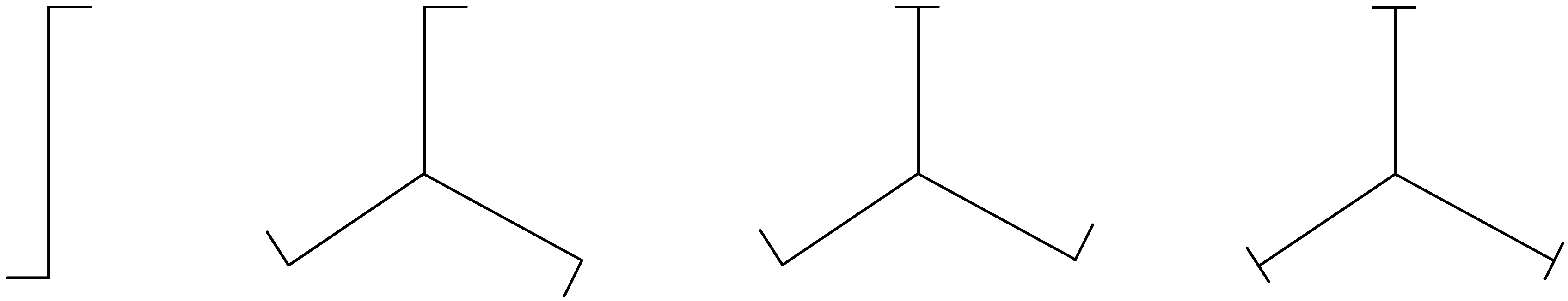
$$D_1,\ D_2,\ D_3,\ \cdots\ .$$

Shape Symmetry Analysis: Some Basics

- The first group is called the **cyclic group** C_n consisting of the repetitions of a single rotation
- The second group is called the **dihedral group** D_n which is a group of these rotations combined with reflections.
- These are the only possible central symmetries in two-dimension

Shape Symmetry Analysis: Some Basics

- Examples of symmetries: (a) C_2-symmetry; (b) C_3-symmetry; (c) D_1-symmetry; (d) D_3-symmetry.



(a) (b) (c) (d)

- Since exact symmetry rarely occurs in real images, **fuzzy symmetry**, here, **describes the approximate or imperfect symmetry of irregular shapes**

Continuous Bilateral Symmetry Measure

- We perform a series of transformations, reflection and rotation operations of a graph
- These transformation operations are used to construct an **Abelian group**, ***H***
- The group *H* has
  - elements: certain motions selected from the totality of these operations; a binary operation on this set is "followed by".

- Based on the definition the group, the concept of **bilateral central symmetry degree** is proposed, leading to the definition of a new shape descriptor, the **bilateral symmetricity**.

Continuous Bilateral Symmetry Measure: Construction of the Abelian Group

- For a given graph M:
  - The origin of the Cartesian coordinate system is set at its centroid.
  - Let *n* be a positive number, *r* be an anticlockwise rotation of M through $2\pi/n$ about the z-axis passing through the origin O and perpendicular to the x-O-y plane. Then,

$$I, r, r^2, \dots, r^{n\text{-}1} \; (r^n = I)$$

represents *n*-fold anticlockwise rotations of M about the centroid (origin O) through

Continuous Bilateral Symmetry Measure: Construction of the Abelian Group

$$\alpha_i = 2\pi i/n\,, \quad i = 0, 1, 2, \ldots, n\text{-}1$$

respectively

- each of which leaves the system unchanged in form;
- *I* represents the identity operation

- This sequence of rotation operations constitute a cyclic group

$$C_n = \left\{ I, r, r^2, \ldots, r^{n\text{-}1} \right\}$$

Continuous Bilateral Symmetry Measure: Construction of the Abelian Group

- The symbol f_{II} denotes a pair of mirror reflection operation of M about the x-axis as mirror line. Thus,

$$f_{\mathrm{II}} = f^2 = I$$

- We construct another group *F* consisting of the single element based on the reflection operation

Continuous Bilateral Symmetry Measure: Construction of the Abelian Group

- **PROPOSITION 1**

  Consider the set H

$$H: I, rf_{\mathrm{II}}, r^2 f_{\mathrm{II}}, \ldots, r^{(n\text{-}1)} f_{\mathrm{II}} \qquad (r^n f_{\mathrm{II}} = I)$$

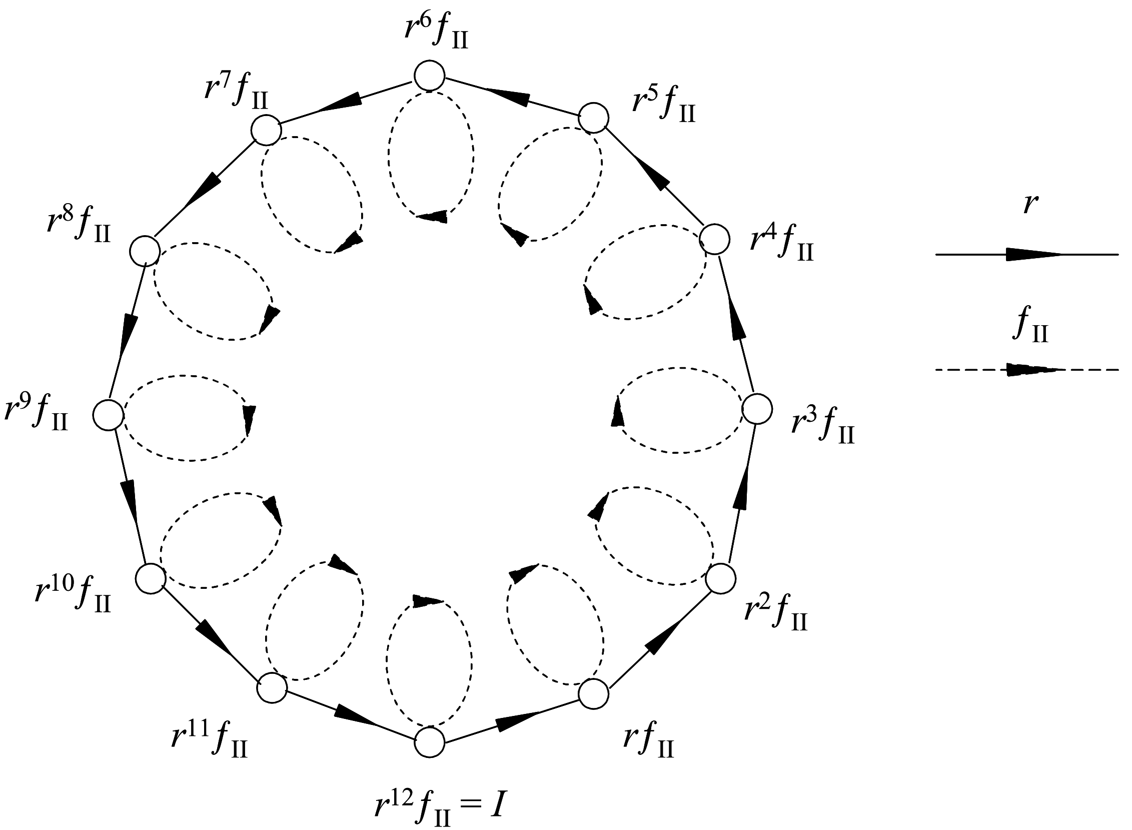
  H is a finite Abelian group of order n

- ***LEMMA 1***

  H is not a dihedral group

Continuous Bilateral Symmetry Measure: Cayley Diagram of the Group *H*

- illustration when n = 12



Continuous Bilateral Symmetry Measure: Cayley Diagram of the Group *H*

- A ***word*** is defined as a finite sequence of generators and their inverses

- Each *word* representing an element in the group can be interpreted as a path or a specific sequence of directed segments of the Cayley diagram.

- The graph of the group *H* is a connected network, that is, there are paths from each vertex to every other vertex

Continuous Bilateral Symmetry Measure: Bilateral Central Symmetry Degree

- **DEFINITION 1** *(Bilateral central symmetry degree)*

  Let M *be a graph in the Euclidean space* R^2.

  All possible transformations of M *including rotation operations 'followed by' a pair of reflection operations are denoted by H, which is a direct product of groups* C_n *and F:*

$$H = C_n \times F$$

Continuous Bilateral Symmetry Measure: Bilateral Central Symmetry Degree

- **DEFINITION 1.** *(Cont'd)*

  Let $\mathrm{W}: w_i\ ,\quad i = 0,1,2,\ldots,n-1$ *be a set of words on all elements of group H.*

  The group H with generators r and f_{II} *is denoted by*

  $$\mathrm{gp}\{r, f_{\mathrm{II}}\}$$

  Let M' *be a transformed graph from the original graph* M *after i^th rotation operations, which corresponds to word* $w'_i = r^i$

  Let M" *be a transformed graph from graph* M' *after one reflection operation, which corresponds to a sequence of operations, denoted by* $w''_i = r^i \sqrt{f_{\mathrm{II}}} = r^i f$

Continuous Bilateral Symmetry Measure: Bilateral Central Symmetry Degree

- **DEFINITION 1.** *(Cont'd)*

  Let A be an area

  We define the **bilateral central symmetry degree**, **BCSD(*i*)**, *about x-axis as being the ratio of the area of the intersection of graph* M' *and its transformed graph* M" *to the area of M, that is*

$$\mathrm{BCSD}(i) = \left.\frac{\mathrm{A}(\mathrm{M}_i^{'} \cap \mathrm{M}_i^{''})}{\mathrm{A}(\mathrm{M})}\right|_{\substack{H=\mathrm{gp}\{r,\, f_{\mathrm{II}}\} \\ word\ w_i,\ i = 0,1,2,\ldots,n-1}}$$

Continuous Bilateral Symmetry Measure: Bilateral Symmetricity

- **DEFINITION 2** *(Bilateral symmetricity)*

  …

  We define the **bilateral symmetricity** δ_{B} *of a graph* M *relative to the group H as being the maximum value of the bilateral central symmetry degree with word* w_i

$$\delta_{\mathrm{B}} = \max\{\mathrm{BCSD}(i)\}\Bigg|_{word\ w_i,\ i = 0,1,2,\ldots,n-1}^{H = \mathrm{gp}\{r, f_{\mathrm{II}}\}}$$

Continuous Bilateral Symmetry Measure: Bilateral Symmetricity

- The value of the bilateral symmetricity ranges from 0 to 1
  - $\delta_B \rightarrow 1$, the graph M becomes perfect or crisp bilateral symmetric
  - $\delta_B \rightarrow 0$, the graph M has the lowest fuzzy bilateral symmetry
- The bilateral symmetricity δ_B is translation-, rotation-, reflection- and scaling invariant

Continuous Bilateral Symmetry Measure: Algorithm for computing the bilateral symmetricity

1. Rotate M anticlockwise about centroid by $\alpha_i = 2\pi i/n$, i = 0, 1, 2, …, n-1.
2. Perform a pair of reflection operation about x-axis as a mirror line through the centroid.
3. If index of the operation i < n, calculate the BCSD, then go to step 1, otherwise (if i = n), go to step 4.
4. Calculate δ_B

Continuous Bilateral Symmetry Measure: Bilateral Symmetry Axis

- This definition gives both the location and the number of bilateral symmetry axis

- **DEFINITION 3** *(Bilateral symmetry axis)*

  If the word on group H is $w_i = r^i f_{\mathrm{II}}$, *the x-axis at which the bilateral symmetricity is obtained at word* w_i *is defined as the bilateral symmetry axis of graph* M.

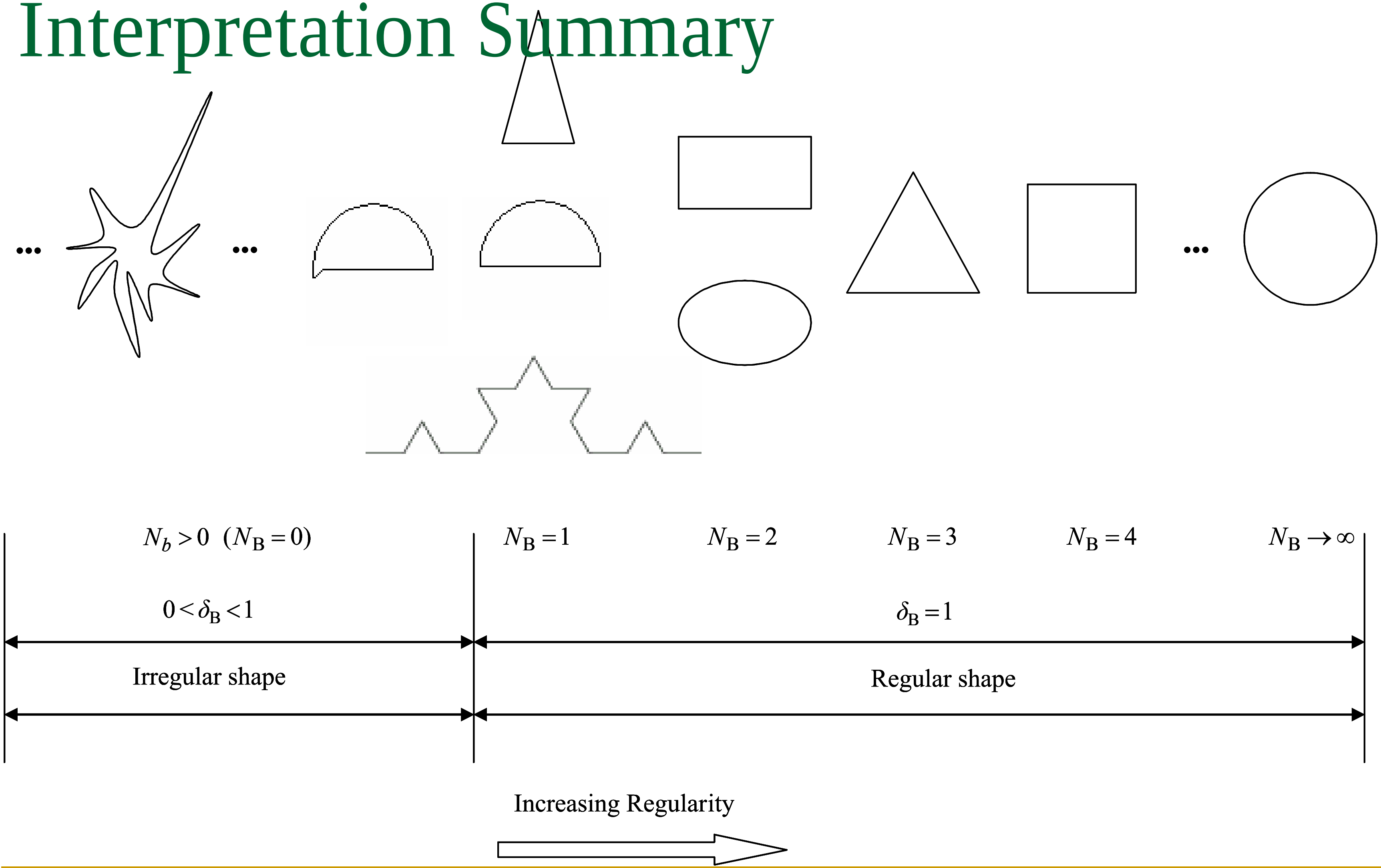
  The number of bilateral symmetry axis is denoted N_b

Continuous Bilateral Symmetry Measure: Bilateral Symmetry Axis

- When the transformed graph M” is in congruence with the transformed graph M’ after *i*th rotation operations, the graph **M becomes perfect bilateral symmetric**,

- Therefore, the bilateral symmetry axis becomes **perfect bilateral symmetry axis**
  - N_B is used to denote the number of perfect bilateral symmetry axis ($\delta_B = 1$)

Continuous Bilateral Symmetry Measure: Interpretation Summary



Continuous Rotational Symmetry Measure:

- We construct a series of rotation transformations without reflection operations for a given graph.

- The **fuzzy rotational symmetry measure** is derived based on these transformations
  - which corresponds to a cyclic group C_n

- This leads to the definitions of the **rotational symmetricity** δ_{R} and the **number of rotational symmetry axis** N_{r}

Continuous Rotational Symmetry Measure: Rotational Central Symmetry Degree

- **DEFINITION 4** *(Rotational central symmetry degree)*

  Let M *be a graph in the Euclidean space* R^2,

  Let M' *be a transformed graph from the original graph M.*

  A sequence of rotation transformations constitute a cyclic group C_n. *Hence, the transformed graph* M' *can be expressed as,*

$$\mathrm{M}' = C_n \cdot \mathrm{M}$$

Continuous Rotational Symmetry Measure: Rotational Central Symmetry Degree

- **DEFINITION 4** *(Cont'd)*

  Let $\mathrm{W} : w_i \,, \quad i = 0, 1, 2, \ldots, n-1$ *be a set of words on all elements of group* C_n *. The group with generators r is denoted by* $\mathrm{gp}\{r\}$*. Let A be an area.*

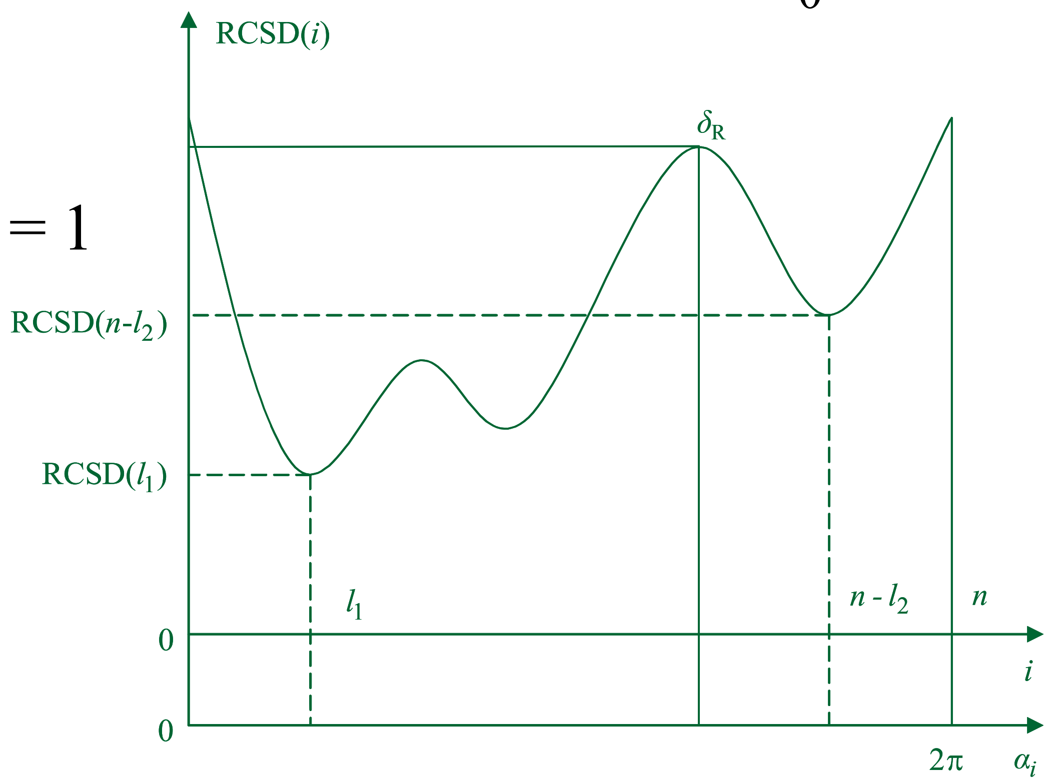
  We define the ***rotational central symmetry degree, RCSD(i),*** *as being the ratio of the area of the intersection of graph* M *and its transformed graph* M' *to the area of M.*

$$\mathrm{RCSD}(i) = \frac{\mathrm{A}(\mathrm{M} \cap \mathrm{M}_i^{'})}{\mathrm{A}(\mathrm{M})} \Bigg|_{\substack{C_n = \mathrm{gp}\{r\} \\ word\ w_i,\ i = 0,1,2,\ldots,n-1}}$$

Continuous Rotational Symmetry Measure: Rotational Symmetricity

- Before defining the rotational symmetricity, we need to discard the RCSD data both at and near $\alpha_0 = 0$ and $\alpha_n = 2\pi$

$$\mathrm{RCSD}(0) = \mathrm{RCSD}(n) = 1$$



Continuous Rotational Symmetry Measure: Rotational Symmetricity

- **DEFINITION 5** *(Rotational symmetricity)*

  Let $\mathrm{W} : w_i \,, \quad i = 0, 1, 2, \ldots, n - 1$ *be a set of words on all elements of group* C_n *. The group with generators r is denoted by* $\mathrm{gp}\{r\}$*.*

  … (l_1 and l_2) …

  We define the ***rotational symmetricity*** δ_{R} *of a graph M relative to the group* C_n *as being the maximum value of the rotational central symmetry degree with word* w_i*,*

$$\delta_{\mathrm{R}} = \max\{\mathrm{RCSD}(i)\} \left| \begin{array}{l} C_n = \mathrm{gp}\{r\} \\ word\ w_i,\ l_1 \le i \le n - l_2 \end{array} \right.$$

Continuous Rotational Symmetry Measure: Rotational Symmetricity

- The value of the rotational symmetricity ranges from 0 to 1
  - $\delta_R \rightarrow 1$, the graph M becomes **perfect rotational symmetric**
  - $\delta_R \rightarrow 0$, the graph M has the **lowest fuzzy rotational symmetry**

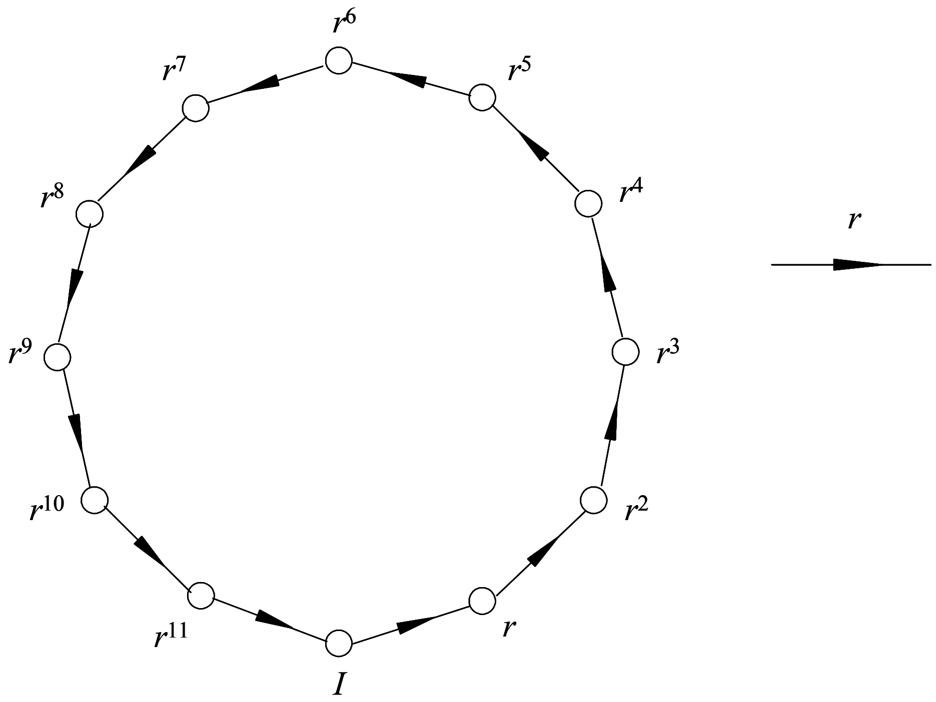
- For a given graph, the rotational symmetricity is independent of the position of the graph
- The rotational symmetricity is translation-, rotation-, reflection- and scaling invariant

Continuous Rotational Symmetry Measure: Algorithm for computing the rotational symmetricity

1. Rotate M anticlockwise about centroid by ,
   i = 0, 1, 2, …, *n*.
2. If index of the operation i ≤ n, calculate the RCSD, then go to step 1, otherwise (if i > n), go to step 3.
3. Find the value of l_1 and l_2, then select RCSD(*i*) data for $l_1 \leq i \leq n - l_2$
4. Compute δ_R

Continuous Rotational Symmetry Measure: Rotational Symmetry axis

- We introduce the fuzzy **rotational symmetry axis** based on the Cayley diagram of constructed group C_n
  - e.g n = 12



Continuous Rotational Symmetry Measure: Rotational Symmetry axis

- **DEFINITION 6** *(Rotational symmetry axis)*
  - *Let* $\mathrm{W} : w_i \,, \quad i = 0, 1, 2, \ldots, n-1$ *be a set of words on all elements of group* C_n *.*
  - *Consider the Cayley diagram of group* C_n *which is an n-gon whose sides are directed segments r.*
  - *If the rotational symmetricity is obtained at word* $w_i = r^i$ *then the* ***rotational symmetry axis*** *is defined as the half line extending from the centre of the n-gon and passing through vertex* w_i*.*
  - *The number of bilateral symmetry axis is denoted* N_r

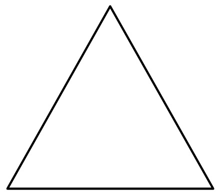
Continuous Rotational Symmetry Measure: Rotational Symmetry axis

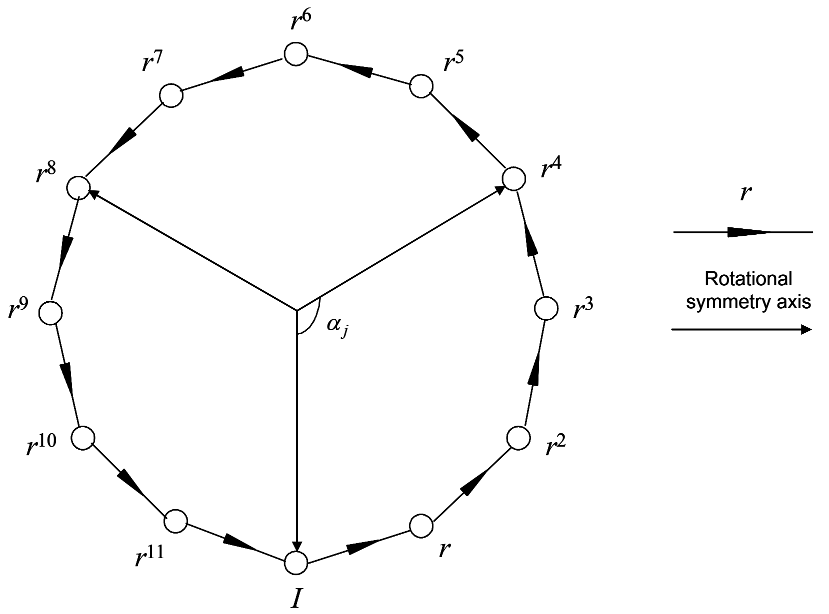
- For regular shapes $\delta_{\mathrm{R}} = 1$
  - the value of N_{R} is counted as the number of congruent motions which brings the shape into coincidence with itself during rotation.
  - use the minimum angle through which the shape has congruent motion as the basic angle of the rotation, α_j where *j* is the corresponding number of rotation

$$N_{\mathrm{R}} = 2\pi/\alpha_j \qquad \alpha_j = 2\pi j/n$$

$$N_{\mathrm{R}} = n / j$$

Continuous Rotational Symmetry Measure: Rotational Symmetry axis





- rotational symmetricity is first obtained at word

$$w_i = w_4 = r^4$$

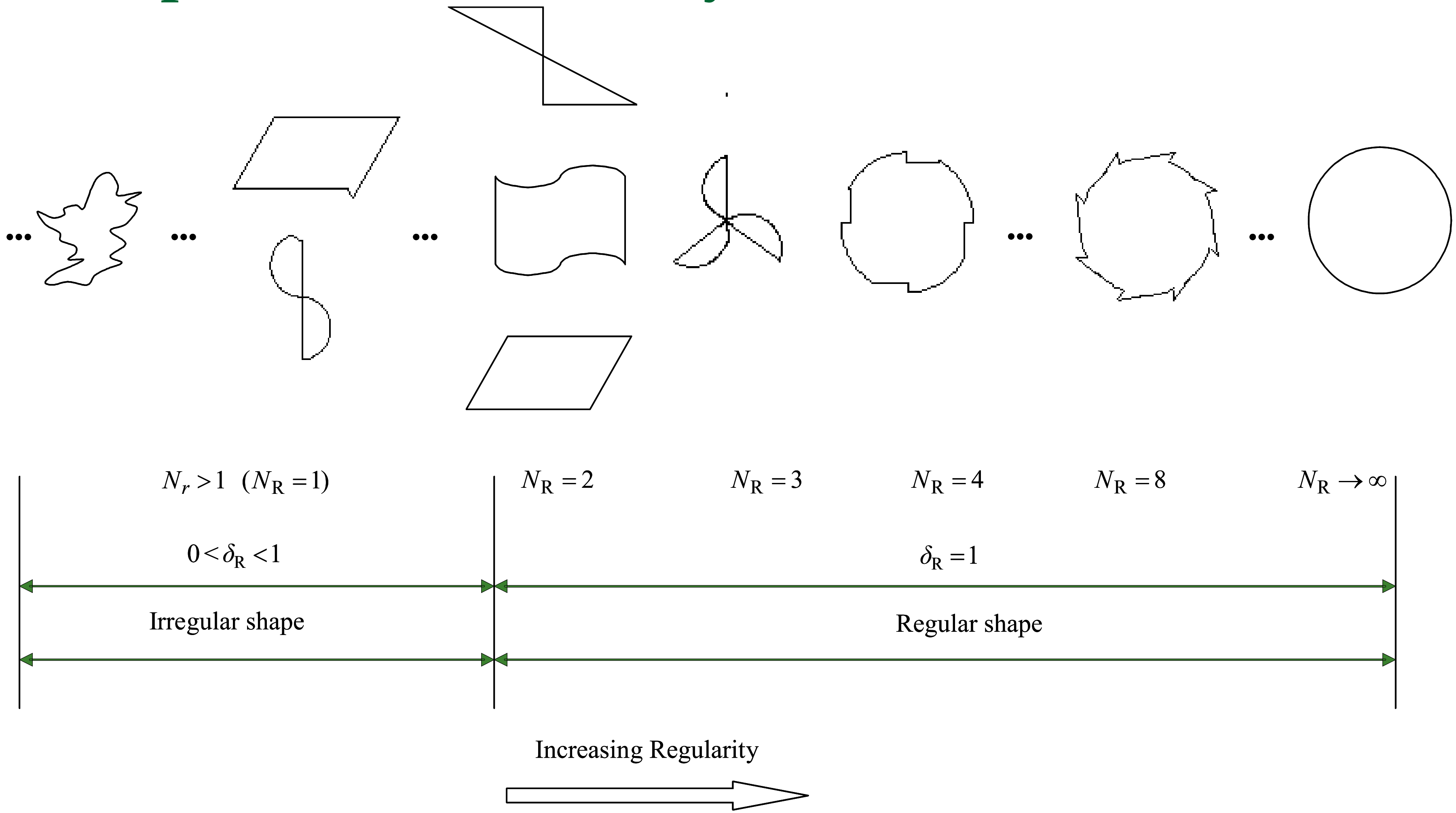
- its basic angle of the rotation

$$\alpha_j = \alpha_4 = 2\pi 4/12 = 2\pi / 3$$

- Hence, the number of perfect rotational symmetry axis

$$N_R = 12/4 = 3$$

Continuous Rotational Symmetry Measure: Interpretation Summary



Optimisation: Searching for the Centre of Symmetry

- So far we have assumed that the best fuzzy symmetry axis for computing symmetricity is near enough to the centroid and to the x-axis.
  - true for some shapes, especially the shape with high regularity
  - for some other shapes, calculating the symmetry measure about a point other than centroid will give a different value of symmetry measure

Optimisation: Searching for the Centre of Symmetry

- **DEFINITION 7**

  Let M *be a graph in Euclidean space R^2. The centre of fuzzy bilateral symmetry of a graph* M*, denoted x_B, is defined as the point about which bilateral symmetricity becomes maximum.*

- **DEFINITION 8**

  Let M *be a graph in Euclidean space R^2. The centre of fuzzy rotational symmetry of a graph* M*, denoted x_R, is defined as the point about which rotational symmetricity becomes maximum.*

  - Note that, in general, the centre of bilateral symmetry may not be in the same position as the centre of rotational symmetry
  - Nelder-Mead method is a common nonlinear optimisation algorithm

Optimisation: Searching for the Centre of Symmetry

(a) (b)

- (a) an octagon shape and its centroid ('○') and centre of symmetry ('×');
- (b) a breast tumour shape and its centroid and centre of symmetry.

Optimisation: Searching for the Centre of Symmetry

- **DEFINITION 9** (*Maximum bilateral symmetricity)*

  Let M *be a graph in Euclidean space* R^2*. The maximum bilateral symmetricity,* δ_{Bmax}*, is defined as the maximum value of bilateral symmetricity about the centre of bilateral symmetry* x_B*,*

$$\delta_{\text{Bmax}} = \max\left(\delta_{\text{B}}\right)$$

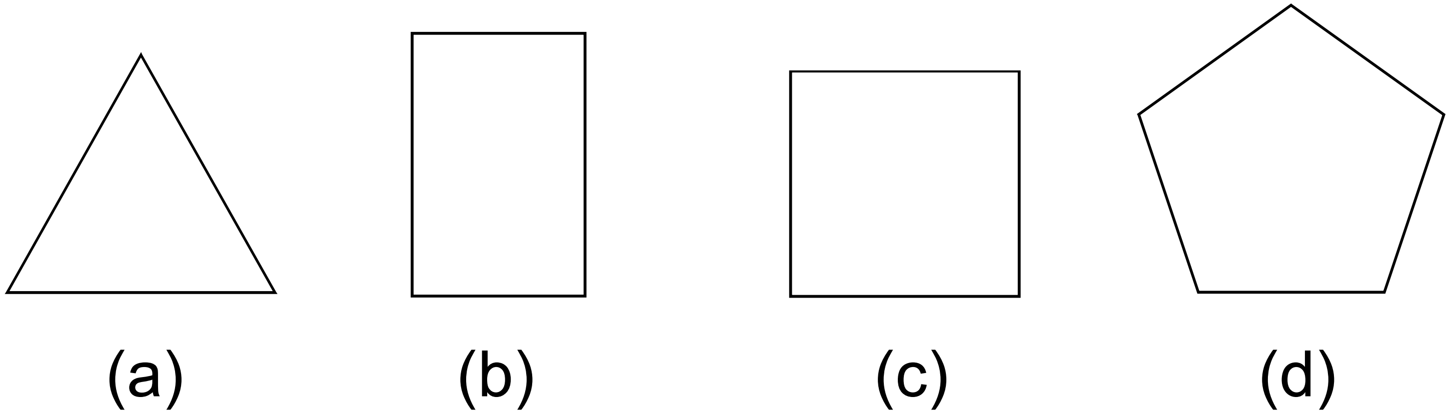
- **DEFINITION 10** (*Maximum rotational symmetricity)*

  Let M *be a graph in Euclidean space* R^2*. The maximum rotational symmetricity,* δ_{Rmax}*, is defined as the maximum value of rotational symmetricity about the centre of rotational symmetry* x_R*.*

$$\delta_{\text{Rmax}} = \max\left(\delta_{\text{R}}\right)$$

Results

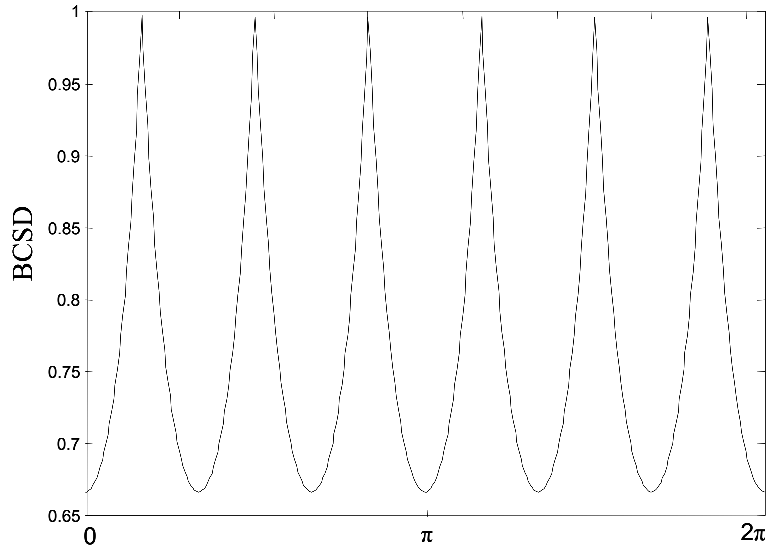
- Typical regular shapes, n =360 $\delta_B = \delta_R = 1$



  - (a) triangle $N_B = N_R = 3$
  - (b) rectangle $N_B = N_R = 2$
  - (c) square $N_B = N_R = 4$
  - (d) pentagon $N_B = N_R = 5$

Results

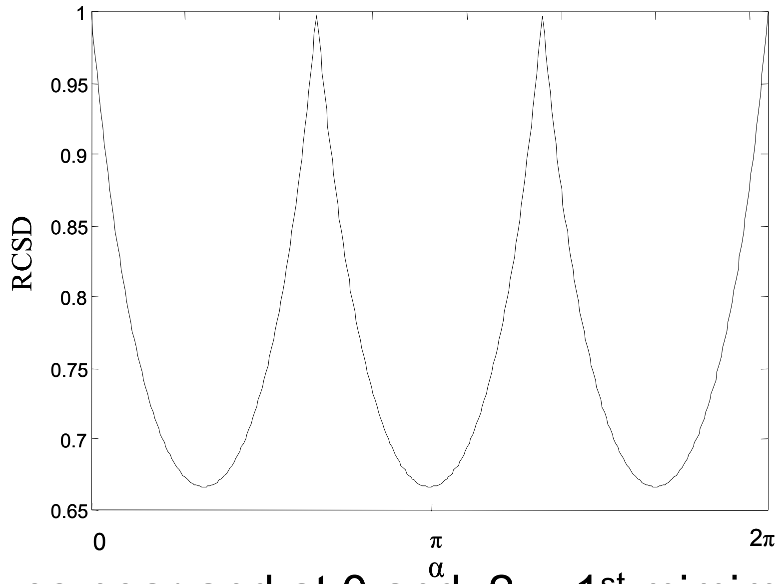
- For illustration: BCSD of the triangle shape (*n* is set as 360)



  - BCSD curves have the period of π
  - N_B : nb of peaks / 2; δ_B : maximum value
  - where there are peaks at 0 and 2π radians, one peak is counted

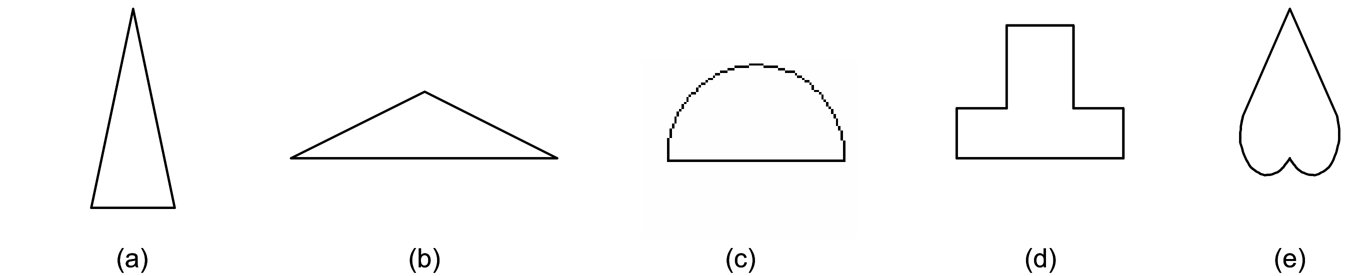
Results

- For illustration: RCSD of the triangle shape (*n* is set as 360)

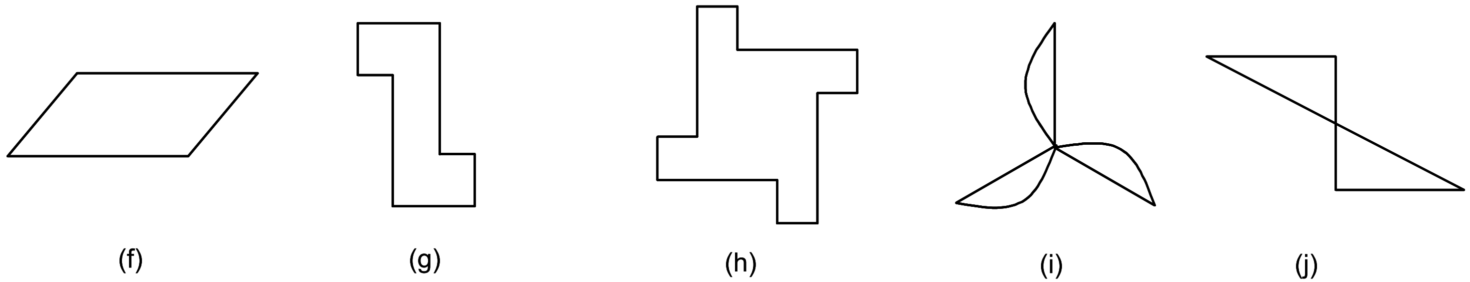


  - disregard values near and at 0 and 2π: 1st mimimum at π/3 and last 5π/3
  - for $\pi/3 < \alpha < 5\pi/3$ N_R : nb of peaks +1, δ_R : maximum value
- Note for regular shapes, $N_B > 0$ and/or $N_R > 1$

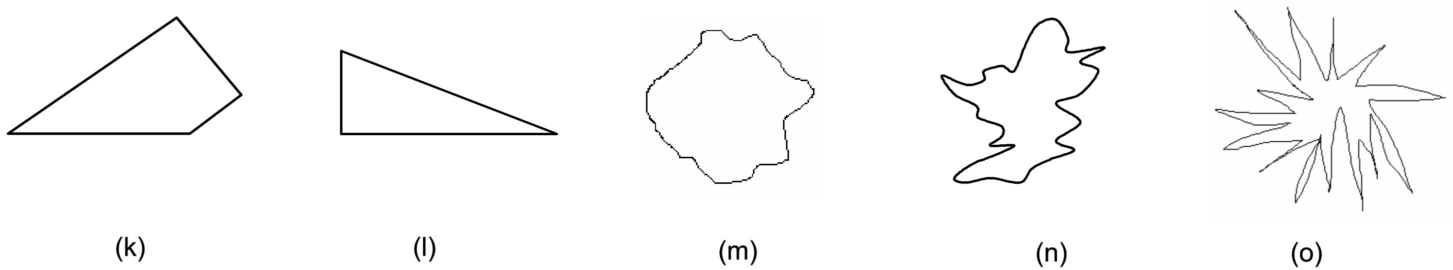
Results



- (a)-(e) perfect bilateral symmetric but not strictly rotational symmetric



- (f)-(j) not perfect bilateral symmetric but perfect rotational symmetric

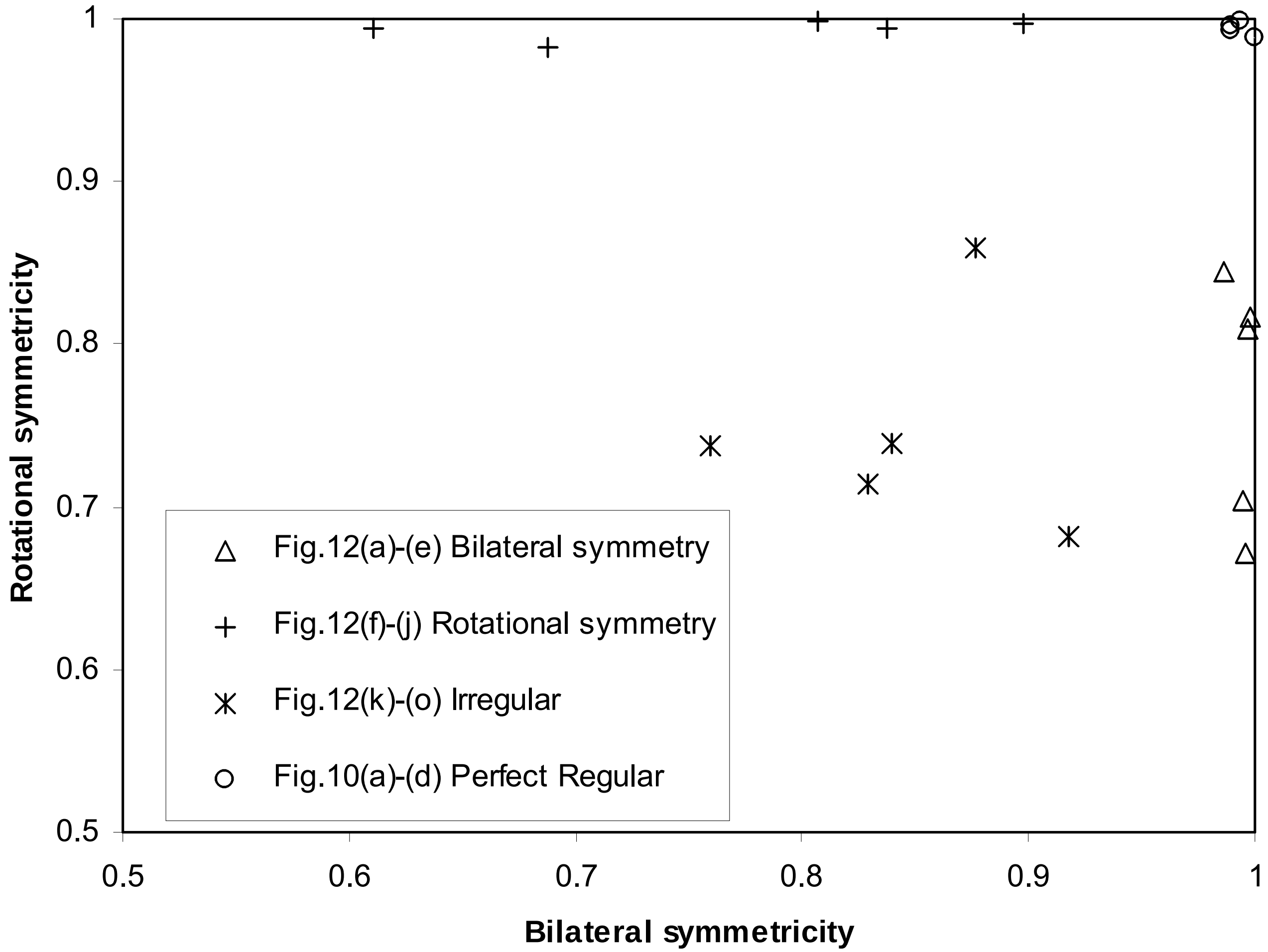


- (k)-(o) neither perfect bilateral symmetric nor perfect rotational symmetric

Results

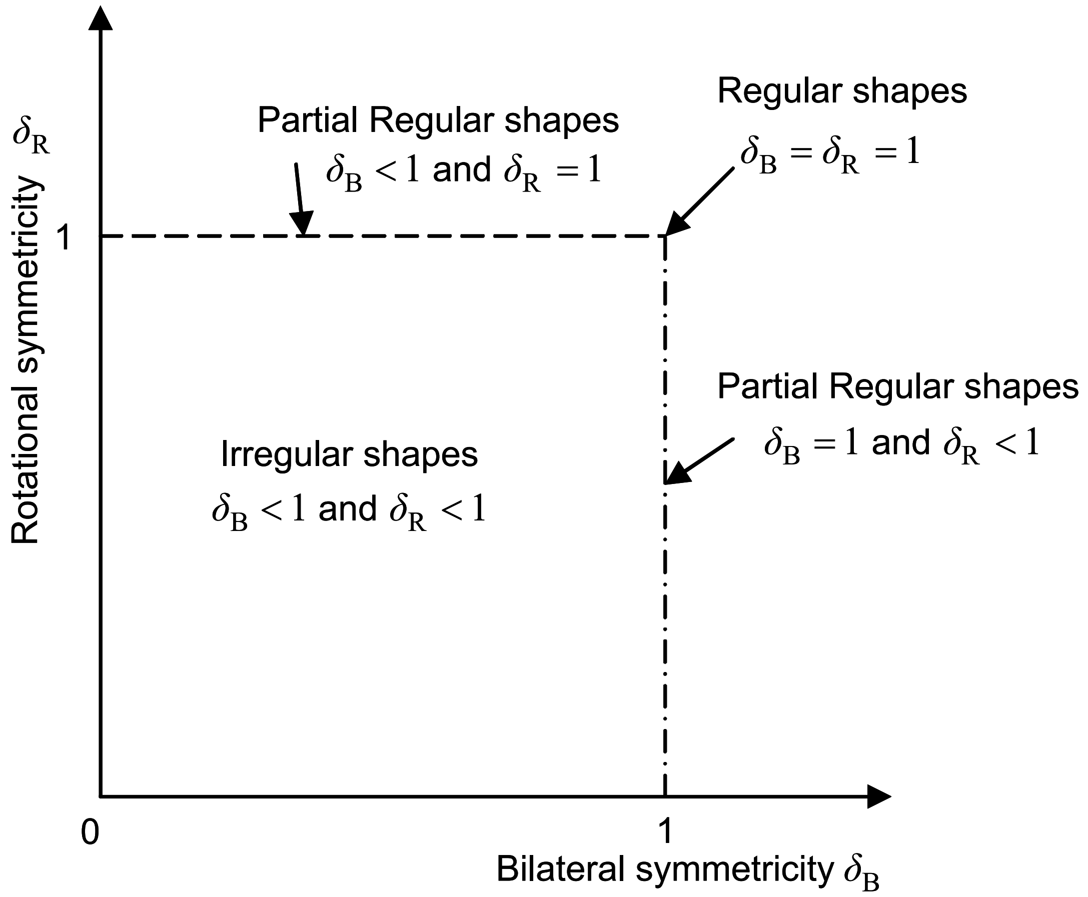
	Shape	δ_B	δ_R	N_B	N_b	N_R	N_r
PBS & FRS	(a)	0.9949	0.7035	1	--	1	3
	(b)	0.9956	0.6717	1	--	1	3
	(c)	0.9979	0.8165	1	--	1	2
	(d)	0.9858	0.8451	1	--	1	3
	(e)	0.9969	0.8097	1	--	1	5
FBS & PRS	(f)	0.8372	0.9939	0	2	2	--
	(g)	0.8065	0.9988	0	2	2	--
	(h)	0.8972	0.9969	0	4	4	--
	(i)	0.6876	0.9831	0	1	3	--
	(j)	0.6112	0.9936	0	1	2	--
FBS & FRS	(k)	0.8392	0.7386	0	1	1	3
	(l)	0.9175	0.6811	0	1	1	4
	(m)	0.8768	0.8592	0	2	1	4
	(n)	0.8288	0.7138	0	1	1	3
	(o)	0.7598	0.7382	0	2	1	4

Results



Rotational symmetricity
Bilateral symmetricity
1
0.9
0.8
0.7
0.6
0.5
0.5
0.6
0.7
0.8
0.9
1
Fig.12(a)-(e) Bilateral symmetry
Fig.12(f)-(j) Rotational symmetry
Fig.12(k)-(o) Irregular
Fig.10(a)-(d) Perfect Regular

Results



δ_R
Rotational symmetricity
1
0
1
Bilateral symmetricity δ_B
Partial Regular shapes
$\delta_B < 1$ and $\delta_R = 1$
Regular shapes
$\delta_B = \delta_R = 1$
Partial Regular shapes
$\delta_B = 1$ and $\delta_R < 1$
Irregular shapes
$\delta_B < 1$ and $\delta_R < 1$

Conclusion

- Symmetry seems to be the last thing one would consider to describe the regularity (or irregularity) of an object with arbitrary shape

- Concept of fuzzy symmetry measure to characterise the irregularity of graphs with arbitrary shapes
  - concept of central symmetry degree
  - concept symmetricity in terms of both fuzzy bilateral symmetry and fuzzy rotational symmetry

- These symmetry measures are based on geometrical operations and contain geometric information of the shape

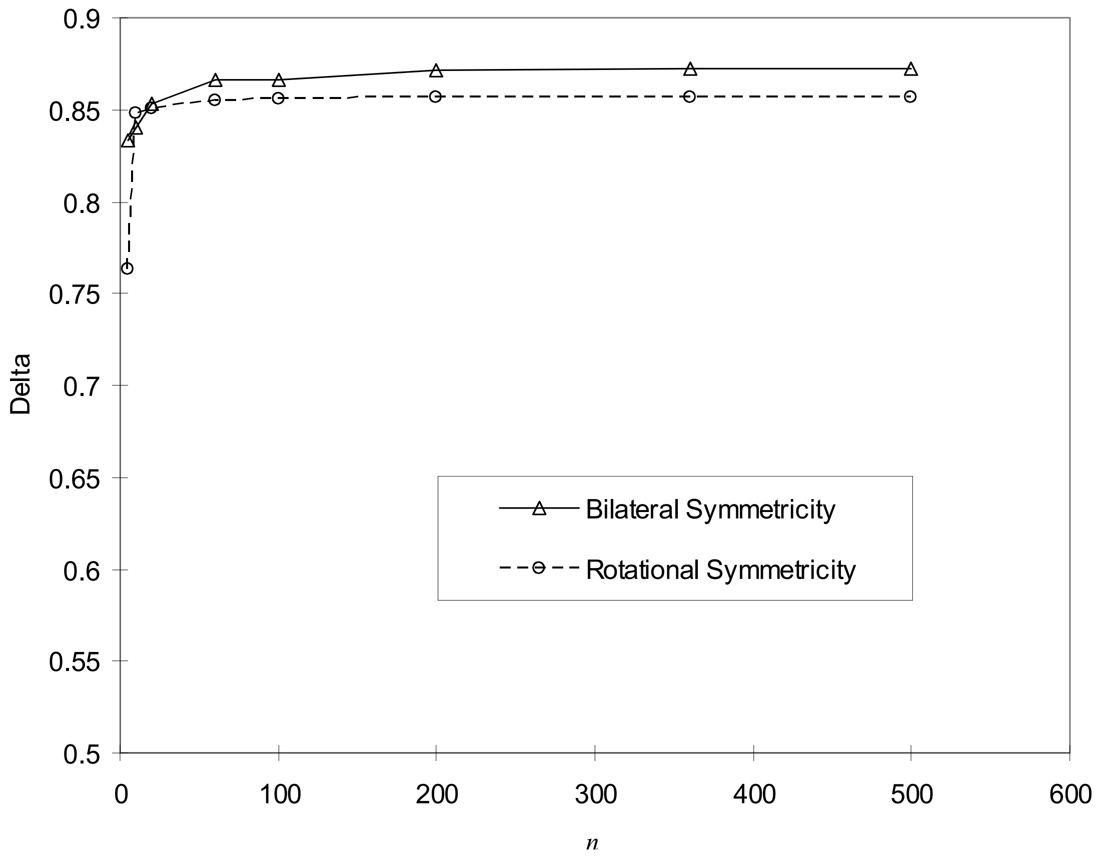
- Concepts of the number of bilateral symmetry axis and rotational symmetry axis

Conclusion

- **THEOREM 1.** *For a regular shape, if the value of bilateral symmetricity and rotational symmetricity both equal to the value of one, that is* $\delta_B = \delta_R = 1$*, then the number of perfect bilateral symmetry axis equal to the number of perfect rotational symmetry axis, that is* $N_B = N_R$.

- **THEOREM 2.** *For a regular shape, if the value of bilateral symmetricity* δ_B *equals the value of one and the number of perfect bilateral symmetry axis* N_B *is greater than one, then, the shape is rotational symmetric, that is* $\delta_R = 1$.
  - Theorem 2 reflects the fact that the number of perfect bilateral symmetry axis characterises the rotationally symmetric property of the regular shapes

Conclusion



- When n ≥ 200
  - average relative error of δ_B due to *n* is less than 0.8%
  - average relative error of δ_R is less than 0.1%